On the Axiom of Reducibility

by Dennis J. Darland

1 The Axiom is given in *12.1 of Principia Mathematica by Whitehead and Russell.

$$(\exists f) : (x) \phi x \equiv f!x$$

What I take to be important here is that ϕ is a function of x only.

2 Types of propositional functions

PM assumes there to be (at least one) individuals of type 0. (types need only be relative.) There are (let us suppose people are individuals - just for illustration). Then general!(Napoleon). is true, and general! is a predicative function of an individual. So general is of type 1. Let f!(ϕ) =df ϕ is a predicate required in a great general. $\phi!$ is of type 1. f! is type 2. if x is any individual, let

$$\text{isagreatgeneral}(x) = \text{df}\,(\phi) : f!(\phi) \supset \phi!x$$

But so defined, isagreatgeneral is not a predicative function. Its definition involves f!, of type 2, and isagreatgeneral is desired to be type 1. A predicative function is one that only has arguments less than its type, and at least one of exactly one less. My belief is now that the axiom of reducibility only says when there is a function of only free variables of type lower than the function then there exists an equivalent predicative function of the same variables. Only predicative functions can be quantified over (indicated by a "!").

isagreatgeneral's definition only has the variable x free, although other expressions occur in its definition. f is a constant.

Russell appied the above function to Napoleon.

[I am calling general! a predicative function (as I think Russell would have. For any given x general!x is true or false. There are multiple values of x for which general!x true or false - but no value of x for which it is both, so, in that way, it is a many-one relation. It would also be a many-one relation if the values are propositions, but the nature of propositions is beyond the scope of this paper.) Call general!, etc. expressions if you want, where I call them predicative functions] I believe Russell thought this prohibited by the theory of types, without the axiom of reducibility. Russell thought the axiom needed for some reasoning otherwise prohibited. I think the theory of types, as Russell used it, all individuals were of type 0. All predicative functions of indiviuduals were type 1. All type 1 predicative functions only had arguments of type 0. All predicative functions of type 2 only had arguments of type 1, or type 0. And so on. This was to avoid paradoxes. But it is too restictive for some mathematical purposes. So, if a non-predicative function (expression) has only free variables of type n or less than n there is a equivalent predicative function of type n + 1. Also it is required that f not be free in ϕ. Otherwise one could have $(\exists f)(x) \sim f!x \equiv$fx

3 Why does PM need the Axiom?

I think the axiom serves multiple purposes. It, even if not impredicative (placing no restrictions on bound predicate variables), some of the theory of classes and relations could be established. But being impredicative it is even more powerful. I think one impredicative use may be for real numbers. Take the property of being a rational number whose squares are less than the rational number 2. One needs such properties of rational numbers to construct real numbers. But such properties may involve totalities similar to the one involving great generals.

4 On How to Read PM

In the first edition, free variables such as x and y, were considered "real" variables for individuals . In the second edition, it was decided that these should be considered apparent variables, and a universal quantifier for the variable placed in front. Letters, "a", "b", were intended to be constants - not variables, in effect part of the propositional schema. And thus not in need of a quantifier.

Functions are indicated by f, g, ϕ, ψ, χ, θ. These are schema, unless used with an exclamation mark, which indicates a variable. I think, even in the first edition, the quantification for such functions was explicit, with "!", for functional variables, and in other cases they inficate schema. When variables, they may take on values for which no linguistic expression corresponds - as, for example, in the cases of some real numbers.

5 The Axiom in General

$$(\exists f)\colon (x_1, x_2, x_3, \ldots, x_n)\phi(x_1, x_2, x_{3,} \ldots, x_n). \equiv f!(x_1, x_2, x_3, \ldots, x_n)$$

Rules

There is some initial collection of predicative functions of any type.

ϕ is constructed from predicative functions, and logical connectives and some quantifiers.

the x_i are the only free variables in ϕ.

the x_i are all of type less than the type of ϕ., and at least one of type one less

the x_i can be predicative functions as well as individuals.

f is not free in ϕ

Such an f can be used to construct more ϕ's.

6 What the Axiom Permits

Consider Russell's example from above: general!(Napoleon). is true, and general! is a predicative function of an individual. So general is of type 1. Let f!(ϕ) =df ϕ is a predicate required in a great general. ϕ! is of type 1. f! is type 2. if x is any individual, let

$$\text{isagreatgeneral}(x) = \text{df}\,(\phi)\colon f!(\phi) \supset \phi! x$$

This is a special instance of:

$$g(f!, \phi!, x)$$

But, as I understand it, this is prohibited by the theory of types, becuse g has arguments that are not all of types lower. That is to say, because predicative functions all must have all arguments type lower, g!, is in general prohibited. But, in the special case of isagreatgeneral, the axiom guarantees that there is an equivalent predicative function, because the only free variable is of type one less.