correct one to be used. Therefore, we will not pursue this further.

There is a way to avoid the cubic algebraic equation derived from the five-term analysis. The idea is to include yet another additional term in the analysis, and see if some cancellation becomes possible. Indeed, this proved feasible, and so we have a "six-term" analysis for solving both the order and radius of convergence for a pair of conjugate poles. To facilitate the writing of the equations, we define two constants

$$CTR := 2\cos(\theta)\ 10^{s} = 2bh/c^{2} ,$$

and $$RCS := 10^{-2s} = (c/h)^{2} .$$

We also define five ratios

$$RM0 := F(m)/F(m-1) ,$$

$$RM1 := F(m-1)/F(m-2) ,$$

$$RM2 := F(m-2)/F(m-3) ,$$

$$RM3 := F(m-3)/F(m-4) ,$$

$$RM4 := F(m-4)/F(m-5) .$$

We divide Eq.[60] by F(m-1) and divide Eq.[61] by F(m-2). When written with the constants and ratios defined above, these equations become

$$(m-1)*RM0 = (m+p-2)*CTR - \frac{1}{RCS}*\frac{(m+2p-3)}{RM1} , \qquad [67]$$

and $$(m-2)*RM1 = (m+p-3)*CTR - \frac{1}{RCS}*\frac{(m+2p-4)}{RM2} . \qquad [68]$$

Subtraction of Eq.[68] from Eq.[67] yields

$$(m-1)*RM0 - (m-2)*RM1 = CTR - \frac{1}{RCS}*\left[\frac{m+2p-3}{RM1} - \frac{m+2p-4}{RM2}\right] . \qquad [69]$$

Next, we repeat Eq.[69] for the next lower index.

$$(m-2)*RM1 - (m-3)*RM2 = CTR - \frac{1}{RCS}*\left[\frac{m+2p-4}{RM2} - \frac{m+2p-5}{RM3}\right] \qquad [70]$$

Equation[70] involves F(m-4), which is in the denominator of RM3. This means that a total of five terms of the Taylor series is used so far. Subtraction of Eq.[70] from Eq.[69] yields

$$(m-1)*RM0 - 2(m-2)*RM1 + (m-3)*RM2 =$$

$$= \frac{1}{RCS}*\left[2\frac{m+2p-4}{RM2} - \frac{m+2p-3}{RM1} - \frac{m+2p-5}{RM3}\right] . \qquad [71]$$

Now, we define some grand constants.

$$NR1 := (m-1)*RM0 - 2(m-2)*RM1 + (m-3)*RM2 ,$$

$$NR2 := (m-2)*RM1 - 2(m-3)*RM2 + (m-4)*RM3 ,$$

$$DR1 := - 1/RM1 + 2/RM2 - 1/RM3 ,$$

$$DR2 := - 1/RM2 + 2/RM3 - 1/RM4 ,$$

$$DS1 := 3/RM1 - 8/RM2 + 5/RM3 ,$$

and $$DS2 := 3/RM2 - 8/RM3 + 5/RM4 .$$

Solving Eq.[71] for the order p, we obtain

$$p = \frac{RCS*NR1 - DS1}{2*DR1} - \frac{m}{2} . \qquad [72]$$

Finally, we repeat Eq.[72] for the next lower index, eliminate p, and solve for RCS. The result is

$$10^{2s} = \frac{1}{RCS} = \frac{NR1*DR2 - NR2*DR1}{DS1*DR2 - DS2*DR1 + DR1*DR2} . \qquad [73]$$

Equation[73] involves all the Taylor series terms from F(m) to F(m-5), thus the name "six-term" analysis.

From Eq.[73], we can obtain the radius of convergence. With a known radius of convergence, we can find the order of the poles from Eq.[72], and the elevation angle from the four-term analysis, Eq.[64]. this six-term analysis is valid for poles of all orders, integer and real, positive and negative, rational and irrational. [The author is indebted to Mr. Manuel Prieto for much of the details in this derivation of the six-term analysis.]

The six-term analysis is not the final word on the study of Taylor series behavior. We have thus far only looked into those functions that have either a singularity on the real axis or a conjugate pair of singularities in the complex plane. In the next few sections, we will study the effects of other functions on the Taylor series behavior. The other functions may be secondary singularities or an analytic function. There is also the need to understand the Taylor expansions of functions that contain an essential singularity.

D. THE EFFECT OF SECONDARY POLES.

The convergence behavior of the Taylor series of functions that have a single pole on the real axis, or a single conjugate pair of poles, was discussed in the previous sections. We have shown that the terms of the Taylor series follow easily recognizable paths in a semi-logarithm plot of the terms versus their index (or order of differentiation). One can quickly glance at such a graph and observe the presence or absence of poles in the function, whose series is plotted. This easy recognition is not always followed by easy calculations. The general behavior of the series terms readily manifest itself graphically, because the graphical information does not have to be accurate for comprehension. It is a different matter when the same data is used for estimating the order and location of the pole, or poles. In this section and the next, we will analyze the disturbance introduced by other functions and secondary poles. A secondary pole in the function is a singularity whose distance from the point of expansion is greater than the radius of convergence.

Figure III-8 below is a graph of $\log|F(m)|$ versus m for conditions similar to that for Figure III-6, with the difference being that there is a conjugate pair of secondary poles. The secondary poles are at an angle of about 167 degrees and at a distance of about 1.5 Rc. When this graph is compared with that in Figure III-6, the effect of the secondary poles is shown clearly by the points marked with different symbols. The first few terms show two effects. First, the phase angle between terms is smaller in the first half-cycle than in all the other half-cycles. This is because the secondary elevation angle is smaller than that for the primary pole. Secondly, there is a separation of alternate terms in the first two half-cycles. This is due to a difference in residues for the primary and secondary poles. It should be noted that the effects of the secondary pole have dissipated by the 30-th term of the series.

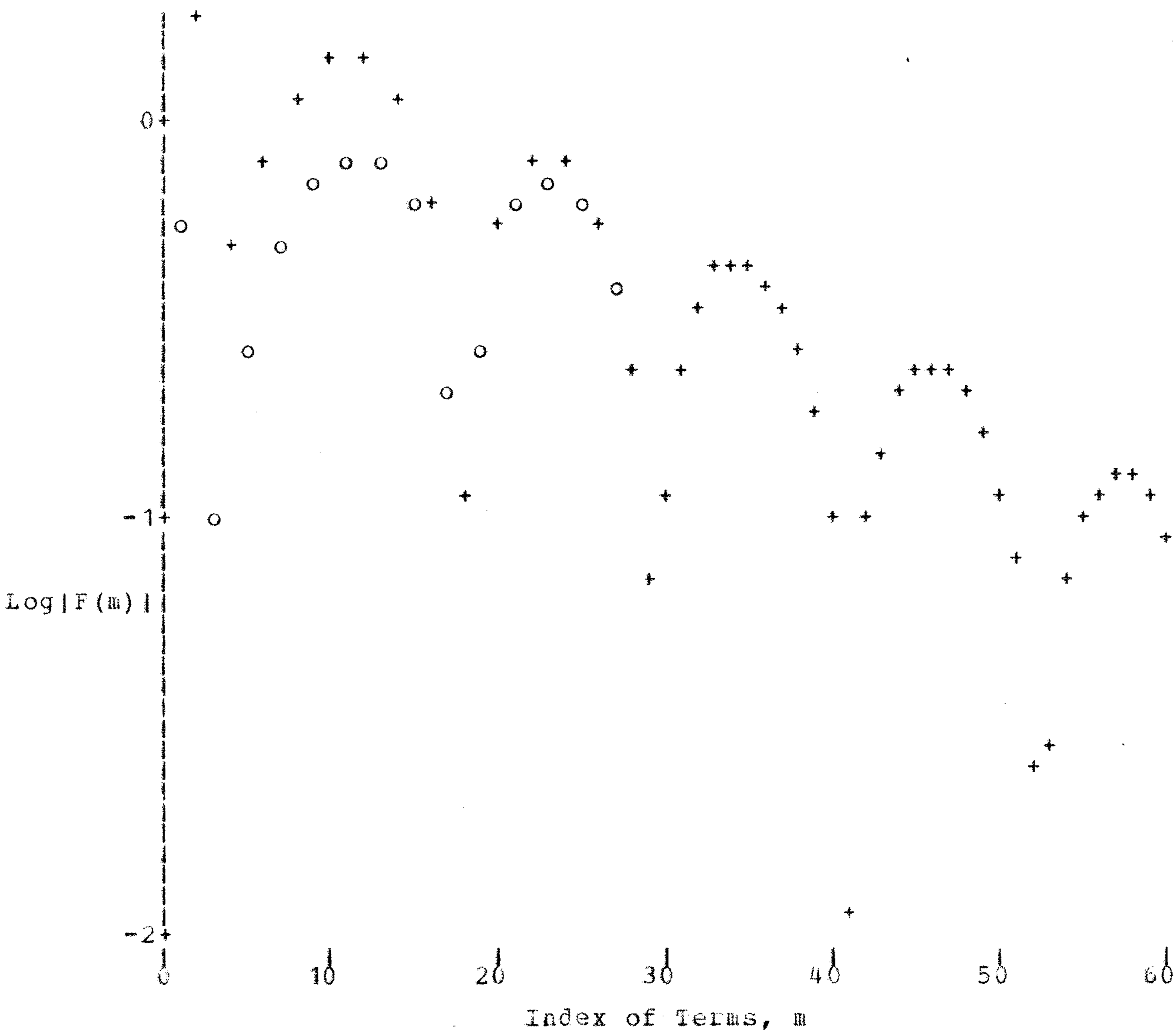

FIGURE III-8. The Effect of Secondary Poles.
(Secondary at 165 degrees, b = 1.5 Rc.)

We will now analyze the effects introduced by other functions and secondary poles. This analysis is concerned with errors in the radius-of-convergence calculation using the two-term and three-term analyses. First, we examine the effect of multiplying the function f(x;10) by an analytic function.

Consider the function

$$f(x) = \frac{g(x)}{(x-a)^{p}} \quad , \qquad [74]$$

where g(x) is an analytic function in x. The Taylor expansion for the function f(x;74) has a recursive relation derivable from Eq.[13].

$$(m-1)F(m)*U(1) + (m+p-2)F(m-1)*U(2) = G(m) \quad . \qquad [75]$$

The recursive relation in Eq.[75] has been disturbed from Eq.[14] by the addition of G(m) on the right-hand side.

This additional term, G(m), is the m-th term of the Taylor expansion for the analytic function, g(x). We will look at the Taylor series for two types of analytic functions, those with a finite length and those with an infinite length. The Taylor series for polynomials are finite in length. Therefore, for some m larger than the degree of the polynomial, G(m)=0. The recursive relation above is then back to its original expression, Eq.[14]. This means that in convergence analysis, we want to use a series length longer than most polynomials encountered. If a specified polynomial is very long, we simply must use an even longer series. Should a truncated polynomial be acceptable as an accurate approximation for the full polynomial, then we will use a truncated polynomial shorter than our Taylor series. Otherwise, we will be solving a wrong problem. Therefore, when the analytic function has a Taylor series of finite length, we can either use a longer series, or truncate the polynomial.

When the analytic function has a Taylor series of infinite length, there will be unavoidable error in the radius of convergence calculation using the two-term or three-term analyses. To study the nature of this error, we return to analyze f(x;74) in a different manner. Consider

$$f(x) = w(x)*g(x) \quad , \qquad [76]$$

where w(x) is the function of the pole

$$w(x) = \frac{1}{(x-a)^p} \; .$$

The m-th term of the F-series is given by

$$F(m) = W(m)*G(1) + W(m-1)*G(2) + \ldots + W(1)*G(m) \; . \qquad [77]$$

Equation[77] is simply the modified Leibnitz rule written out longhand. The relation between W(m) and W(m-1) is given by the two-term analysis, Eq.[18]. The same relation holds for W(m-1)/W(m-2) and all the other adjacent pairs of terms in the W-series. This means that the right-hand side of Eq.[77] can be written in terms of W(m) only.

$$F(m) = W(m)*\left[G(1) + G(2)\frac{m-1}{m+p-2}\left(\frac{a}{h}\right) + \ldots + G(m)\frac{(m-1)!\,(p-1)!}{(m+p-2)!}\left(\frac{a}{h}\right)^{m-1}\right] \qquad [78]$$

Similarly, the expression for F(m-1) is

$$F(m-1) = W(m-1)*\left[G(1) + \ldots + G(m-1)\frac{(m-2)!\,(p-1)!}{(m+p-3)!}\left(\frac{a}{h}\right)^{m-2}\right] \; . \qquad [79]$$

For convenience, we define SUMGM to be the entire sum in Eq.[78], and we define SUMG1 to be the entire sum in Eq.[79]. Then, Eq.[78] and Eq.[79] become

$$F(m) = W(m)*SUMGM \; ,$$

and $$F(m-1) = W(m-1)*SUMG1 \; . \qquad [80]$$

Substitution of Eq.[80] into Eq.[18] leads us to the final result

$$\frac{h}{Rc(est)} = \frac{F(m)}{F(m-1)}*\frac{m-1}{m+p-2} = \frac{W(m)}{W(m-1)}*\frac{m-1}{m+p-2}*\frac{SUMGM}{SUMG1} = \frac{h}{Rc}*\frac{SUMGM}{SUMG1} \; .$$

When compared to the original two-term analysis, Eq.[18], the expression above has a relative error given by

$$\text{relative error} = \frac{SUMGM - SUMG1}{SUMG1} . \qquad [81]$$

The relative error given in Eq.[81] is dependent on the order of the pole p, the length of the Taylor series m, and the function g(x). The error can be analyzed only when the function g(x) is specified.

Consider the function g(x) = exp(-bx) as a representative example of an analytic function with an infinite Taylor series. The exponential function has an infinite series typical of all the analytic functions. Its Taylor series expanded about x=0 with an increment h is given by

$$e^{-b(x+h)} = 1 - b*h + \frac{b^2 h^2}{2!} - \frac{b^3 h^3}{3!} + \frac{b^4 h^4}{4!} - \ldots .$$

The m-th term of this series is

$$G(m) = \frac{(-b*h)^{m-1}}{(m-1)!} .$$

Now, the sum SUMGM becomes

$$SUMGm = \sum_{i=1}^{m} \frac{(m-1)!(m+p-i-1)!}{(m-i)!(m+p-2)!(i-1)!}(-a*b)^{i-1} , \qquad [82]$$

and the sum SUMG1 is given by

$$SUMG1 = \sum_{i=1}^{m-1} \frac{(m-2)!(m+p-i-2)!}{(m-i-1)!(m+p-3)!(i-1)!}(-a*b)^{i-1} . \qquad [83]$$

The relative error in the two-term analysis for any order p with an exponential g(x) can be obtained from the subtitution of Eqs.[82] and [83] into Eq.[81]. Simpler expressions are possible when p is given a value. For example, for p=1, the relative error is

$$\text{relative error} = \frac{\frac{1}{(m-1)!}(-a*b)^{m-1}}{\sum_{i=1}^{m-1} \frac{1}{(i-1)!}(-a*b)^{i-1}} . \qquad [84]$$

This is an important and interesting result. It is the very foundation for the validity of the two-term and four-term analyses when applied to solutions of real-life problems, where analytic functions and secondary poles abound. It is interesting, because this error expression is just the convergence criterion for the Taylor expansion of g(x) if the radius of convergence, a, is identified as the Taylor expansion increment. So, this error is equal to the ratio of the last term of the analytic series divided by the value of the same function, when the expansion increment is the radius of convergence. If the analytic function is properly represented by this series, then the radius of convergence is accurately determined. Basically, what is required is

$$(-Rc*b)^{m-1} < m-1 .$$

If this inequality is not satisfied, there will be error in calculating the radius of convergence. If this inequality is reversed, it will be impossible to calculate the radius of convergence. This latter case is what is refered to as a "stiff" problem. More will be said on this in Chapter VIII.

The conclusion is: the accuracy of the radius-of-convergence calculation is dependent on the convergence rate of the Taylor expansion of the analytic function using an increment equal to the radius of convergence. From a practical point of view, the meaning of this conclusion is that we should use as long a Taylor series as possible in order to be certain that the series of analytic functions have converged with very small remainder terms. This speaks for using very long series lengths; however, the computation time for the solution of real-life problems increases as the square of the series length. So, we must find some compromise. This question will be discussed in the next Chapter. The choice is to use 30-term Taylor series as the standard.

The relative error in the calculation of the radius of convergence for other values of p is considerably larger. For p=2, it is

$$\text{relative error} = \frac{\displaystyle\sum_{i=1}^{m} \frac{1}{m\,(i-2)!}(-Rc*b)^{i-1}}{\displaystyle\sum_{i=1}^{m-1} \frac{m-i}{(i-1)!}(-Rc*b)^{i-1}} \quad . \qquad [85]$$

This relative error is several orders of magnitude larger than that for p=1, Eq.[84]. Similarly, as we will show in Figure III-10, the relative error for p=0 is also orders of magnitude larger than the error for p=1. Therefore, before calculating for the radius of convergence, one should adjust the indices of the Taylor series to change the order of the pole to be close to p=1. This adjustment, or shift in the indices of the series terms, can be accomplished by either

$$V(m-1) = F(m)*(m-1)/h \ ,$$

or

$$V(m+1) = F(m)*h/m \ , \qquad [86]$$

where the V-series is the new series, and F(m) is the original series. This shift in indices can be performed as often as necessary to change the order of the pole to be close to p=1. The purpose of this shift is to take advantage of the fact that the radius of convergence calculation is much more accurate when the order of the pole is close to one.

We now analyze the effect of secondary poles in the same manner as above. Consider the function g(x) to be a function with a pole on the real axis, a "secondary" pole,

$$g(x) = \frac{1}{(x-b)^{p}} \ .$$

The m-th term of the Taylor series for this function, at x=0 with increment h, is derivable from Eq.[18].

$$G(m) = \frac{(m+p-2)!}{(m-1)!\,(p-1)!}\left(\frac{h}{b}\right)^{m-1}(-b)^{-p} \ . \qquad [87]$$

The sum SUMGM becomes

$$SUMGM = \sum_{i=1}^{m} \frac{(m-1)!\,(m+p-i-1)!\,(i+p-2)!}{(m-i)!\,(m+p-2)!\,(p-1)!\,(i-1)!}(Rc/b)^{i-1} \ , \qquad (88)$$

and the sum SUMG1 becomes

$$\text{SUMG1} = \sum_{i=1}^{m-1} \frac{(m-2)!\,(m+p-i-2)!\,(i+p-2)!}{(m-i-1)!\,(m+p-3)!\,(p-1)!\,(i-1)!} (Rc/b)^{i-1} . \qquad (89)$$

The relative error in the two-term analysis for order p with a secondary pole can be obtained by substituting Eqs.[88] and [89] into Eq.[81]. For p=1, the ratios of factorials in all three equations above are equal to one. Therefore, the relative error is

$$\text{relative error} = \frac{(Rc/b)^{m-1}}{\sum_{i=1}^{m-1} (Rc/b)^{i-1}} . \qquad [90]$$

The factors in Eq.[90] are identical to the factor in Eq.[87]. So, the relative error in radius of convergence is exactly equal to the error in the Taylor approximation of g(x). Therefore, the same conclusion given above is valid for secondary poles as well. The accuracy of the radius-of-convergence calculation is dependent on the convergence rate of the Taylor expansion of the secondary pole using an increment equal to the radius of convergence.

The relative error for a primary pole of order p=2 is much larger than that for p=1. The expression is given by

$$\text{relative error} = \frac{\sum_{i=1}^{m} \frac{i(i-1)}{m} (Rc/b)^{i-1}}{\sum_{i=1}^{m-1} i(m-i)(Rc/b)^{i-1}} . \qquad [91]$$

Once again, the error in the two-term analysis is orders of magnitude larger when the order of the pole is p=2. Therefore, it is advisable to shift the Taylor series to change the order close to p=1. Some graphs will be shown later to illustrate the behavior of the error under different circumstances.

The error in the radius of convergence calculation using the three-term analysis is obtainable from a similar development. First, we need to define a sum SUMG2 just like those sums in Eq.[80].

$$F(m-2) = W(m-2)*SUMG2$$

We substitute Eq.[80] and the above into Eq.[23] and find

$$(m-1)\frac{W(m)*SUMGM}{W(m-1)*SUMG1} - (m-2)\frac{W(m-1)*SUMG1}{W(m-2)*SUMG2} = \frac{h}{Rc(est)}$$

Since W(x) is the primary-pole function, we can apply Eq.[18] to the two ratios W(m)/W(m-1) and W(m-1)/W(m-2) above. After some messy algebra, this yields

$$\frac{h}{Rc}*\left[(m+p-2)\frac{SUMGM}{SUMG1} + (m+p-3)\frac{SUMG1}{SUMG2}\right] = \frac{h}{Rc(est)} .$$

An expression for the relative error can be obtained for fixed values of the order p. For p=1, the result is

$$\text{relative error} = -(m-2)\frac{(Rc/b)^{m-2}}{\sum_{i=1}^{m-2}(Rc/b)^{i-1}} + (m-1)\frac{(Rc/b)^{m-1}}{\sum_{i=1}^{m-1}(Rc/b)^{i-1}} . \quad [92]$$

As we will see in the graphs to be shown below, the error as given in the above expression is much larger than that for the two-term analysis. This suggests that one should use the two-term analysis except when the three-term analysis is absolutely necessary. The two-term analysis can be used to solve for an unknown order p by shifting the series indices and iterations. This will be discussed further in a later section.

Figure III-9 below is a plot of errors in the calculation of Rc by the two-term and three-term analyses when there is a secondary pole at x=b. The vertical axis is the absolute value of the error. The horizontal axis is the ratio of b/Rc, the distance to the secondary pole divided by the radius of convergence. As expected, the error decreases as the secondary pole is moved further away from the point of expansion. The order of both primary and secondary poles is p=1 in this study, and the length of the Taylor series is 30-terms. Errors in the two-term analysis are marked by "+"; while errors in the three-term analysis are marked by "x". As mentioned earlier, the radius-of-convergence calculation is more accurate with two-term analysis than with three-term analysis. The errors in three-term analysis are about two to three times larger than those in two-term analysis. Also plotted in Figure III-9 is a line marked by "2". The errors indicated by that line are those for primary and secondary poles of order two. These errors are orders of magnitude larger than those for p=1.

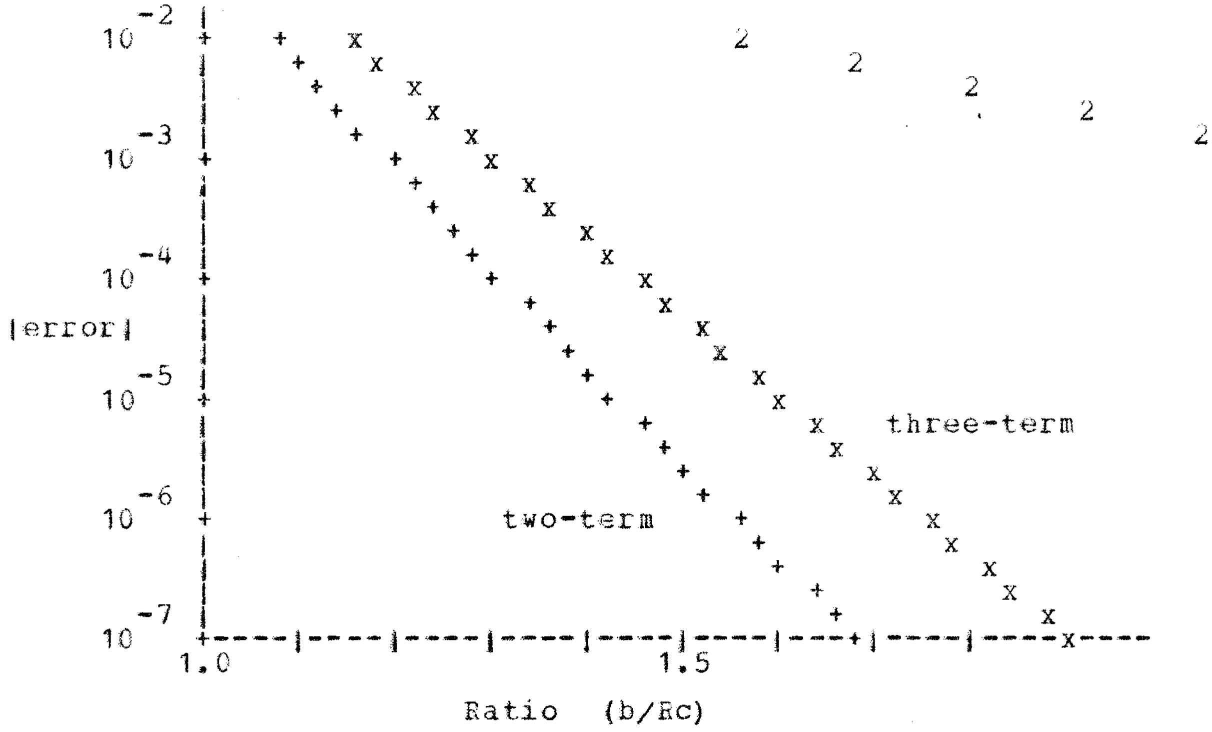

FIGURE III-9. Errors in Rc, with Secondary Pole at x=b.
(30-term Taylor series used)

It is evident from this graph that, for p=1, the accuracy is better than 4-decimal figures when the distance to the secondary pole is larger than 1.3 times the radius of convergence. Even when the secondary pole is as close as 1.1 times Rc, the accuracy is 2-decimal figures. This is evidence for the validity and usefulness of the two-term analysis for Taylor series with secondary poles, as in a real-life problem, when the order is p=1.

Figure III-10 is a plot of the errors in the radius-of-convergence calculation using the two-term analysis for various orders of the poles.

the orders of the poles are varied from p=0 to p=2. There is a sharp minimum in the error when the order of the poles is p=1. Although the error is much higher for orders other than p=1, the calculation for the radius of convergence is still quite accurate. For an order of p=1/2, the error is less than 0.08 percent. However, for an order of p=2, the error is almost ten times larger at 0.5 percent. Therefore, one should use the shifting technique of Eq.[86] to obtain an order of the poles in the neighborhood of p=0 to p=1.

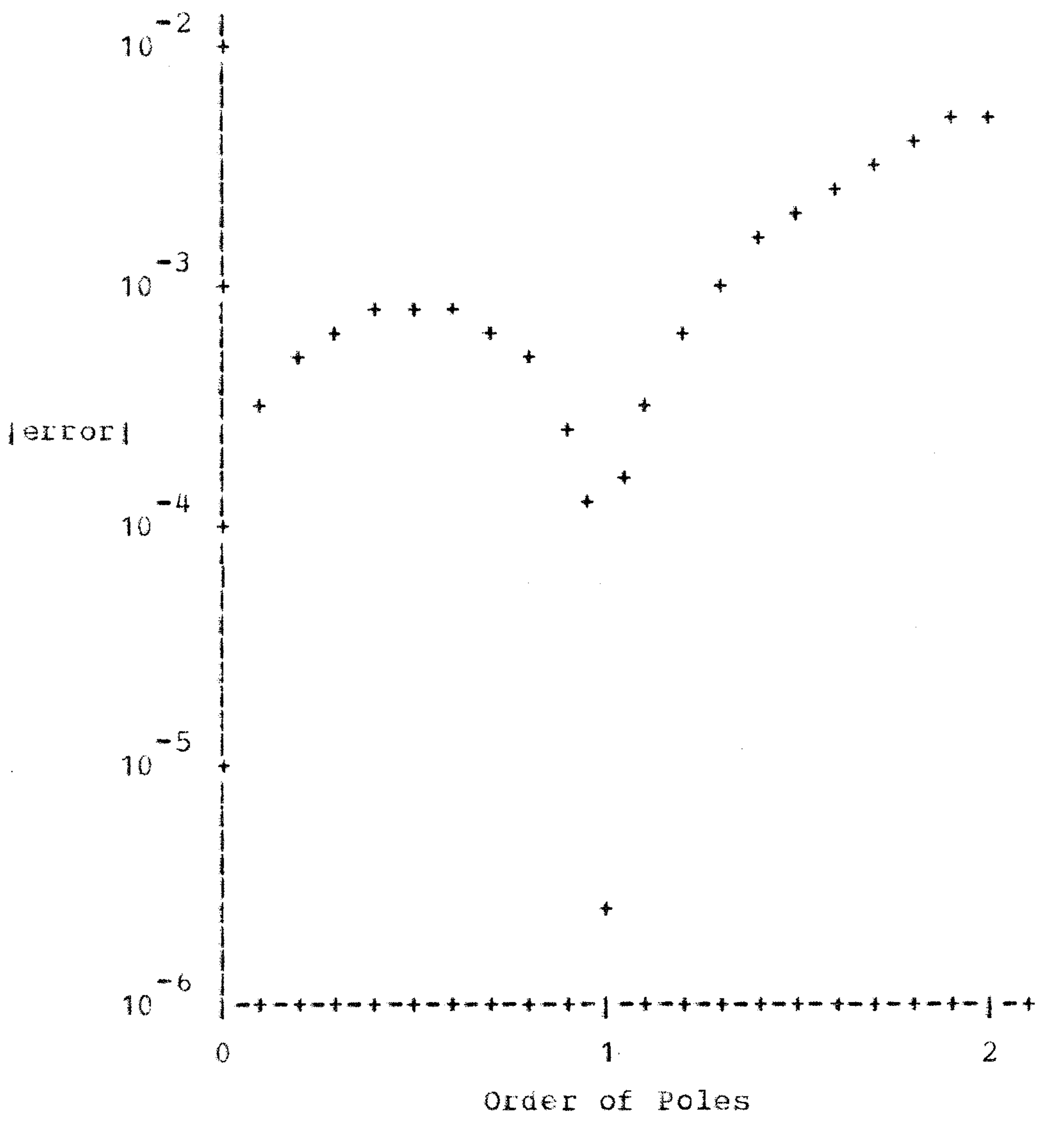

FIGURE III-10. Errors in Rc, with b=1.5Rc, for Orders p=0 to p=2. (30-term Taylor series used)