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THE AUTOMATIC TAYLOR SERIES (ATS) METHOD OF ANALYSIS

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PREFACE

The aim of this book is to describe the development of a method of mathematical analysis in reasonable detail and with sufficient examples to help the reader to an understanding of the Automatic Taylor Series (ATS) method so that he will be able to apply the analysis to his own problems. The author has assumed some acquaintance of Advanced Calculus and a working knowledge of FORTRAN on the part of the reader. Contents of this book are quite straightforward and should provide comfortable reading for the average scientist and/or engineer. Rather than mathematically rigorous theorems, the author uses simple heuristic discussions to show the way to the important ideas behind the Automatic Taylor Series (ATS) method of analysis. He leaves the rigorous proofs to his mathematician colleagues.

The Automatic Taylor Series (ATS) method of mathematical analysis is based on the premise that differentiation of analytic functions can be performed at a specified point by simple additions and multiplications. Once this fact is fully understood, expansion of even the most complicated function into a long Taylor series can be obtained with ease. The power of the Automatic Taylor Series (ATS) method as applied to numerical analysis lies in the accuracy that is obtainable. The availability of very long Taylor series has led to new understanding of the convergence properties of the series in relation to the location of singularities and to the radius of convergence. It is possible to set an error limit beforehand and maintain local error control over the computations within that limit, up to almost machine accuracy.

This book is the result of many years of research on applications of Taylor series to diverse problems in physics and mathematics. The basic ideas of automatic sequential differentiation and the solution of boundary-value problems in linear ordinary differential equations were first presented at an open seminar on May 12, 1963 at Purdue University. This research began in December 1962 when a student (James Williams) and the author investigated the error produced in the numerical solutions of ordinary differential equations using limited-order Taylor series and Runge-Kutta methods. It was found that Taylor series and Runge-Kutta

methods of like order produced comparable errors. It was further found that the order of the numerical method was the prime determining factor for the magnitude of the error. The sixth-order Taylor series method gave errors approximately one order of magnitude less than that produced by a fourth-order Taylor series or a fourth-order Runge-Kutta method. The possibility of developing a variable-order Taylor series method was then investigated. A variable-order Taylor series is a series where all the derivative terms of the series are evaluated easily in a very long series. Then, the error is dependent only on computer limitations and the location of singularities in the solution function. In 1968, the author discovered the relation between the behavior of the Taylor series terms and the location of simple poles in the function being expanded. Thus began the concept of a preset error control. The years since that time have been spent looking into the solutions of a large variety of problems:- solutions of polynomials, nonlinear ordinary differential equations, linear partial differential equations, and nonlinear partial differential equations.

This book is divided into ten Chapters. Chapter I is the general introduction. The procedures for the automatic sequential differentiation of analytic functions are in Chapter II. In Chapter III, the convergence properties of long power series are analyzed and related to the locations of singularities. The subsequent Chapters are devoted to discussions of applications of the (ATS) analysis to specific classes of problems. We begin with simple integrals and linear problems, and run a full gamut to the solutions of nonlinear partial differential equations.

Chronologically, the analyses of quadrature (Chapter V) and linear ordinary differential equations (Chapter VII) were completed in 1963. The development of automatic sequential differentiation including partial differentiation (Chapter II) was completed in 1965. The analyses of long Taylor series (Chapter III) and of optimum series length (Chapter IV) were completed in 1969. The solution of nonlinear ordinary differential equations including the VFCCPL compiler (Chapter VIII) was completed in 1972. The application of the ATS method to polynomials (Chapter VI) was completed in 1974. The application of the ATS method

to linear and non-linear partial differential equations (Chapters IX and X) began in 1973, and is still continuing.

It is with great pleasure that the author acknowledges the contributions of the many persons and organizations that led to the completion of this work. He is grateful to Miss Dorothea Bamberser for introducing him to the digital computer in 1962. Dr. James Williams did the initial research on the errors in numerical solutions of differential equations. Dr. S. D. Conte gave the author encouragement in the years (1963-66). Dr. Ralph Thomas and Professor Ottis Recharad gave the author support and encouragement in the middle years (1970-72). Professors David Colton, George Corliss, and Robert Krueser gave invaluable assistance to the author in the later years (1973-78).

The author is indebted to all the following organizations for their support: Purdue University, University of California at Los Angeles, Battelle Memorial Institute, The Washington State University, Indiana University-Bloomington, and the University of Nebraska-Lincoln. Finally the author owes much to those persons who have read the manuscript and made many helpful suggestions. They include: Professors George Corliss, Robert Krueser, and all those captive students in Numerical Analysis. The author is particularly indebted to Professor George Corliss.

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TABLE OF SYMBOLS

The equations and formulae in this book are written (in draft copy) in a FORTRAN-like format. Some of the odd-looking symbols are explained below.

$f(x)$ = a function of x .

$F(n)$ = the n -th Taylor series term (reduced derivative) of the function $f(x)$.

$\frac{df}{dx}$ = the ordinary derivative of $f(x)$ with respect to x .

$f^{(n)} = \frac{d^n f}{dx^n}$ = the n -th derivative of $f(x)$ with respect to x .

$\frac{\partial f}{\partial x}$ = the partial derivative of $f(x)$ with respect to x .

$\frac{\partial^n f}{\partial x^n}$ = the n -th partial derivative of $f(x)$ with respect to x .

$\int f dx$ = the indefinite integral of f with respect to x .

$\int_a^b f dx$ = the definite integral of f with respect to x , between limits a and b .

π = the value of $\pi = 3.4159265$, etc.

$\sum_{n=1}^{\infty} s(n)$ = the sum from 1 to infinity of the array $s(n)$.

$\sum_{n=2}^k s(n)$ = the sum from 2 to k of the array $s(n)$.

$\prod_{n=1}^k s(n)$ = the product $s(1)s(2)s(3)\dots s(k)$.

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