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                RRRRR            A   A         F
                R    R          AAAAAAA
       DDDDD    RRRRR          A       A
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THE AUTOMATIC TAYLOR SERIES (ATS) METHOD OF ANALYSIS

Y. F. Chang

Computer Science Department

University of Nebraska

Lincoln, Nebraska

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PREFACE

The aim of this book is to describe the development of a method of mathematical analysis in reasonable detail and with sufficient examples to help the reader to an understanding of the Automatic Taylor Series (ATS) method so that he will be able to apply the analysis to his own problems. The author has assumed some acquaintance of Advanced Calculus and a working knowledge of FORTRAN on the part of the reader. Contents of this book are quite straightforward and should provide comfortable reading for the average scientist and/or engineer. Rather than mathematically rigorous theorems, the author uses simple heuristic discussions to show the way to the important ideas behind the Automatic Taylor Series (ATS) method of analysis. He leaves the rigorous proofs to his mathematician colleagues.

The Automatic Taylor Series (ATS) method of mathematical analysis is based on the premise that differentiation of analytic functions can be performed at a specified point by simple additions and multiplications. Once this fact is fully understood, expansion of even the most complicated function into a long Taylor series can be obtained with ease. The power of the Automatic Taylor Series (ATS) method as applied to numerical analysis lies in the accuracy that is obtainable. The availability of very long Taylor series has led to new understanding of the convergence property of the series in relation to the location of singularities and to the radius of convergence. It is possible to set an error limit beforehand and maintain local error control over the computations within that limit, up to almost machine accuracy.

This book is the result of many years of research on applications of Taylor series to diverse problems in physics and mathematics. The basic ideas of automatic sequential differentiation and the solution of boundary-value problems in linear ordinary differential equations were first presented at an open seminar on May 12, 1963 at Purdue University. This research began in December 1962 when a student (James Williams) and the author investigated the error produced in the numerical solutions of ordinary differential equations using limited-order Taylor series and Runge-Kutta methods. It was found that Taylor series and Runge-Kutta methods of like order produced comparable errors. It was further found that the order of the numerical method was the prime determining factor for the magnitude of the error. The sixth-order Taylor series method gave errors approximately one order of magnitude less than that produced by a fourth-order Taylor series or a fourth-order Runge-Kutta method. The possibility of developing a variable-order Taylor series method was then investigated. A variable-order Taylor series is a series where all the derivative terms of the series are evaluated easily in a very long series. Then, the error is dependent only on computer limitations and the location of singularities in the solution function. In 1968, the author discovered the relation between the behavior of the Taylor series terms and the location of simple poles in the function being expanded. Thus began the concept of a preset error control. The years since that time have been spent looking into the solutions of a large variety of problems:- solutions of polynomials, nonlinear ordinary differential equations, linear partial differential equations, and nonlinear partial differential equations.

This book is divided into ten Chapters. Chapter I is the general introduction. The proceedures for the automatic sequential differentiation of analytic functions are in Chapter II. In Chapter III, the convergence properties of long power series are analyzed and related to the locations of singularities. The subsequent Chapters are devoted to discussions of applications of the (ATS) analysis to specific classes of problems. We begin with simple integrals and linear problems, and run a full gamut to the solutions of nonlinear partial differential equations.

Chronologically, the analyses of quadrature (Chapter V) and linear ordinary differential equations (Chapter VII) were completed in 1963. The development of automatic sequential differentiation including partial differentiation (Chapter II) was completed in 1965. The analyses of long Taylor series (Chapter III) and of optimum series length (Chapter IV) were completed in 1969. The solution of nonlinear ordinary differential equations including the YFCCPL compiler (Chapter VIII) was completed in 1972. The application of the ATS method to polynomials (Chapter VI) was completed in 1974. The application of the ATS method to linear and non-linear partial differential equations (Chapters IX and X) began in 1973, and is still continuing.

It is with great pleasure that the author acknowledges the contributions of the many persons and organizations that led to the completion of this work. He is grateful to Miss Dorothea Bamberger for introducing him to the digital computer in 1962. Dr. James Williams did the initial research on the errors in numerical solutions of differential equations. Dr. S. D. Conte gave the author encouragement in the years (1963-66). Dr. Ralph Thomas and Professor Ottis Rechard gave the author support and encouragement in the middle years (1970-72). Professors David Colton, George Corliss, and Robert Krueger gave invaluable assistance to the author in the later years (1973-78).

The author is indebted to all the following organizations for their support: Purdue University, University of California at Los Angeles, Battelle Memorial Institute, The Washington State University, Indiana University-Bloomington, and the University of Nebraska-Lincoln. Finally the author owes much to those persons who have read the manuscript and made many helpful suggestions. They include: Professors George Corliss, Robert Krueger, and all those captive students in Numerical Analysis. The author is particularly indebted to Professor George Corliss.

TABLE OF CONTENTS

ITEM	SUBJECT	Page
PREFACE		i
TABLE OF CONTENTS		iv
TABLE OF SYMBOLS		vi
TABLE OF FIGURES		vii
TABLE OF PROGRAMS		ix
TABLE OF TABLES		xi
CHAPTER I.	INTRODUCTION	1
	A. Reduced Derivatives and Modified Leibnitz Rule.	
CHAPTER II.	AUTOMATIC SEQUENTIAL DIFFERENTIATION	10
	A. Differentiation of Simple Functions and Operations.	
	B. Sequential Differentiation of Complicated Functions.	
	C. L'Hospital Rule.	
	D. Automatic Differentiation of Inverse Functions.	
	E. Partial Differentiations.	
	F. Differentiation of Implicit Functions.	
CHAPTER III.	ANALYSIS OF LONG TAYLOR SERIES	68
	A. Pole on the Real Axis.	
	B. The Order of Poles.	
	C. Poles in the Complex Plane.	
	D. The Effect of Secondary Poles.	
	E. The Last-Pole Situation.	
	F. Essential Singularities.	
	G. Concept of Error Control.	
CHAPTER IV.	OPTINUM SERIES LENGTH AND PROGRAMMING	130
	A. Optimum Series Length.	
	B. Optimum Problem Programming.	
	C. Implicit Functions.	
CHAPTER V.	QUADRATURE	170
	A. Integration of Simple Functions.	
	B. Integration in the Neighborhood of Singularities.	
	C. The ATS Method in Interpolations.	

CHAPTER VI. SOLUTIONS OF POLYNOMIALS AND ALGEBRAIC EQUATIONS . . . 220
A. The Inverse Methods.
B. The Reciprocal Method.
C. Solution of Polynomials (The RTPOLY Program).
D. Solution of Nonlinear Algebraic Equations.

CHAPTER VII. LINEAR ORDINARY DIFFERENTIAL EQUATIONS 270
A. Initial-Value Problems (Non-Numeric Solutions).
B. Boundary-Value Problems (Non-Numeric Solutions).
C. Examples.

CHAPTER VIII. NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS 334
A. Initial-Value Problems.
B. The Order of Poles.
C. Stiff Equations.
D. Two-Point Boundary-Value Problems.
E. Compound Boundary-Value Problems.
F. Eigenvalue Problems.
G. The YFCCPL Compiler.

CHAPTER IX. LINEAR PARTIAL DIFFERENTIAL EQUATIONS 380
A. The Recursive Relation.
B. A Three-Dimensional Solution.
C. Two-Dimensional Solution by Rows or Columns.
D. Separable Variables, Constant Multipliers for Rows.
E. Solution of a Parabolic PDE.
F. Solution of a Tricomi-Type Equation.

CHAPTER X. NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS 430
A. The Recursive Relation.
B. Problems in Analytic Continuation.
C. The Order of Poles.
D. Solutions of PDE's with Two Arbitrary Boundaries.
E. Examples.

APPENDIX A. RECURSIVE RELATIONS OF SIMPLE FUNCTIONS AND OPERATIONS 480
APPENDIX B. MORE STUFF 530
APPENDIX C. ALSO 580
APPENDIX D. WHATEVER 600
INDEX 620

TABLE OF SYMBOLS

The equations and formulae in this book are written (in draft copy) in a FORTRAN-like format. Some of the odd-looking symbols are explained below.

f(x)	= a function of x.
F(n)	= the n-th Taylor series term (reduced derivative) of the function f(x).
$\frac{df}{dx}$	= the ordinary derivative of f(x) with respect to x.
$f^{(n)} = \frac{d^n f}{dx^n}$	= the n-th derivative of f(x) with respect to x.
$\frac{@f}{@x}$	= the partial derivative of f(x) with respect to x.
$\frac{@^n f}{@x^n}$	= the n-th partial derivative of f(x) with respect to x.
(I f dx)	= the indefinite integral of f with respect to x.
(I f dx)$_0$	= the definite integral of f with respect to x, between limits and .
pi	= the value of pi = 3.4159265, etc.
$\text{sum.}_{n=1}^{00}\ s(n)$	= the sum from 1 to infinity of the array s(n).
$\text{sum.}_{n=2}^{k}\ s(n)$	= the sum from 2 to k of the array s(n).
$\text{prod.}_{n=1}^{k}\ s(n)$	= the product s(1)s(2)s(3).....s(k).

TABLE OF FIGURES

ITEM	TITLE	Page
FIGURE II-1.	Building Blocks in the Evaluation of Inverse Derivatives, F(m,n)	46
FIGURE II-2.	Evaluation of the Inverse Sine Function	49
FIGURE III-1.	Closed Contour C in the Complex Plane	69
FIGURE III-2.	Pole Location and Radius of Convergence	70
FIGURE III-3.	Graph of Taylor Terms for a Pole on Real Axis . . .	73
FIGURE III-4.	Graph of Taylor Terms for Poles on Real Axis	77
FIGURE III-5.	A Pair of Conjugate Poles	80
FIGURE III-6.	Graph of Taylor Terms for a Conjugate Pair of Poles (Angle θ = 7.5-degrees, h = .96 Rc.)	88
FIGURE III-7.	Graph of Taylor Terms for a Conjugate Pair of Poles (Angle θ = 74-degrees, h = .95 Rc.)	89
FIGURE III-8.	The Effect of Secondary Poles	97
FIGURE III-9.	Errors in Rc, with Secondary Pole at x=b	105
FIGURE III-10.	Errors in Rc, with |b|=1.5Rc, for p=-1 to p=2 . . .	106

TABLE OF PROGRAMS

The programs in this book are written for the CDC-computer. These programs will usually fail when run on the IBM-computer.

ITEM	NAME OF PROGRAM	Page
PROG. II-1.	The ATS Function	13
PROG. II-2.	The ATT Function	16
PROG. II-3.	Differentiation of f(x) = x**sin(ax)	19
PROG. II-4.	Bessel Function Derivatives (Direct Method)	21
PROG. II-5.	Bessel Function Derivatives (Indirect Method)	24
PROG. II-6.	Differentiation of the Error Function, erf(x)	27
PROG. II-7.	The Error Function, erf(y) with y = sin(ax)	29
PROG. II-8.	The Sine Integral, si(x)	33
PROG. II-9.	The Sine Integral, si(y) with y = erf(x).	35
PROG. II-10.	Function with an Indeterminate Form	40
PROG. II-11.	The ATN-Function.	44
PROG. II-12.	The ATI Subroutine, Version 1	47
PROG. II-13.	The Efficient ATI Subroutine	48
PROG. II-14.	Differentiation of an Inverse Bessel Function	50
PROG. II-15.	The ATP Function	53
PROG. II-16.	The ATQ Function	55
PROG. II-17.	The ATC Function	60
PROG. II-18.	The ATCS Subroutine	60
PROG. II-19.	The Partial Derivatives of f(x,y;151)	62
PROG. II-20.	Partial Derivatives of f(x,y;152)	64
PROG. II-21.	Partial Derivatives of f(x,y;153)	67

TABLE OF TABLES

ITEM	TITLE	Page
TABLE I.	The Values of the Bessel Derivatives	118
TABLE II.		
TABLE III.		
TABLE VI.		