APPENDIX A.

RECURSIVE RELATIONS OF SIMPLE FUNCTIONS AND OPERATIONS

Tabulated below are sets of FORTRAN statements for the sequential differentiation of the operations and functions given. Each set of FORTRAN statements can be embedded inside some larger program for the solution of whatever problem that contains the operation or function involved. Each set of FORTRAN statements can be separated into an initializing part and a recursive part. The recursive part is to be placed within a DO-loop for the generation of successive Taylor series terms. The initializing part must be placed before the DO-loop mentioned above. When several functions and/or operations occur together in a problem, the recursive parts of all the functions can be combined together within the DO-loop, and the initializing parts are placed together before the DO-loop.

The constants and variables used in the table below are defined as follows.

x	is the independent variable.
f and g	are the functions being differentiated. (We use 30-term Taylor series.)
K KA = K - 1 , KB = K - 2 .	is the order of the Taylor series term, and
H	is the increment. Its magnitude should be controled to avoid overflow or underflow.
C and C1	are real constants.
y and y1	are dependent variables (full 30-terms).
xa and xb	are dependent variables (only 2-terms).
A1, A2, and A3	are auxiliary variables (full 30-terms).

The dependent variables xa and xb have only 2-terms in their Taylor series, because they are simple linear functions of the independent x. Examples are:- xa = C+x, xa = C*x, xa = x/C, xb = x+xa, etc. Note that xa = x/C is a linear function of x, while C/x is not a linear function of x.

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   FUNCTION                                   PROGRAM STATEMENTS
---------------------------------------------------------------------------------

1. f = xa*xb . . . . . . . . F(1) = XA(1)*XB(1)
                             F(2) = XA()*XB(2) + XA(2)*XB(1)
                             F(3) = XA(2)*XB(2)

                            DO-LOOP
                             F(K) = 0.

2. f = xa/xb . . . . . . . . F(1) = XA(1)/XB(1)
                             F(2) = (XA(2) - F(1)*XB(2))/XB(1)

                            DO-LOOP
                             F(K) = - F(KA)*XB(2)/XB(1)

3. f = C/xa  . . . . . . . . F(1) = C/XA(1)

                            DO-LOOP
                             F(K) = - F(KA)*XA(2)/XA(1)

           C
4. f = xa , real C   . . . . F(1) = XA(1)**C

                            DO-LOOP
                             F(K) = (C-KB)*XA(2)*F(KA)/XA(1)/KA

            C
4a. f = xa , integer C . . . F(1) = XA(1)**C

                            DO-LOOP
                  For K<C+2  F(K) = (C-KB)*XA(2)*F(KA)/XA(1)/KA
                  For K>C-3  F(K) = 0.

        xa
5. f = C     . . . . . . . . F(1) = C**XA(1)
                             C1 = ALOG(C)

                            DO-LOOP
                             F(K) = C1*F(KA)*XA(2)/KA

         xb
6. f = xa    . . . . . . . . F(1) = XA(1)**XB(1)
                             A1(1) = ALOG(XA(1))
                             A1(2) = XA(2)/XA(1)
                             A2(1) = (A1(1)*XB(2)+A1(2)*XB(1))/H
                             F(2) = F(1)*A2(1)*H

                            DO-LOOP
                             A1(K) = - A1(KA)*XA(2)*KB/XA(1)/KA
                             A2(KA) = (A1(KA)*XB(2)+A1(K)*XB(1))*KA/H
                             F(K) = ATS(KA,F,A2,1)*H/KA

7. f = xa*y  . . . . . . . . F(1) = XA(1)*Y(1)

                            DO-LOOP
                             F(K) = Y(KA)*XA(2) + Y(K)*XA(1)

8. f = xa/y  . . . . . . . . F(1) = XA(1)/Y(1)
                             F(2) = (XA(2) - F(1)*Y(2))/Y(1)

                            DO-LOOP
                             F(K) = - ATS(K,Y,F,2)/Y(1)

9. f = y/xa  . . . . . . . . F(1) = Y(1)/XA(1)

                            DO-LOOP
                             F(K) = (Y(K) - F(KA)*XA(2))/XA(1)
```

FUNCTION . . . . PROGRAM STATEMENTS

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10. $f = C/y$

```
F(1) = C/Y(1)

DO-LOOP
  F(K) = - ATS(K,Y,F,2)/Y(1)
```

11. $f = y*y1$

```
F(1) = Y(1)*Y1(1)

DO-LOOP
  F(K) = ATS(K,Y,Y1,1)
```

12. $f = y/y1$

```
F(1) = Y(1)/Y1(1)

DO-LOOP
  F(K) = (Y(K) - ATS(K,Y1,F,2))/Y1(1)
```

13. $f = y^{C}$

```
F(1) = Y(1)**C
F(2) = C*F(1)*Y(2)/Y(1)

DO-LOOP
  F(K) = (C*ATT(KA,F,Y,1)-ATT(KA,Y,F,2))/Y(1)
```

14. $f = C^{y}$

```
F(1) = C**Y(1)
C1 = ALOG(C)

DO-LOOP
  F(K) = ATT(KA,F,Y,1)*C1
```

15. $f = y^{xa}$

```
F(1) = Y(1)**XA(1)
A1(1) = ALOG(Y(1))
A1(2) = Y(2)/Y(1)
A2(1) = (A1(1)*XA(2) + A1(2)*XA(1))/H
F(2) = F(1)*A2(1)*H

DO-LOOP
  A1(K) = (Y(K) - ATT(KA,Y,A1,2))/Y(1)
  A2(KA) = (A1(K)*XA(1)+A1(KA)*XA(2))*KA/H
  F(K) = ATS(KA,F,A2,1)*H/KA
```

16. $f = xa^{y}$

```
F(1) = XA(1)**Y(1)
A1(1) = ALOG(XA(1))
A1(2) = XA(2)/XA(1)
A2(1) = (A1(1)*Y(2) + A1(2)*Y(1))/H
F(2) = F(1)*A2(1)*H

DO-LOOP
  A1(K) = - A1(KA)*XA(2)*KB/XA(1)/KA
  A2(KA) = ATS(K,Y,A1,1)*KA/H
  F(K) = ATS(KA,F,A2,1)*H/KA
```

17. $f = y^{y}$

```
F(1) = Y(1)**Y1(1)
A1(1) = ALOG(Y(1))

DO-LOOP
  A1(K) = (Y(K) - ATT(KA,Y,A1,2))/Y(1)
  A2(KA) = ATS(K,Y,A1,1)*KA/H
  F(K) = ATS(KA,F,A2,1)*H/KA
```

18. $f = (xa)^{1/2}$

```
F(1) = SQRT(XA(1))
F(2) = XA(2)/F(1)/2.

DO-LOOP
  F(K) = 0.
  F(K) = - ATS(K,F,F,2)/F(1)/2.
```

FUNCTION	PROGRAM STATEMENTS
19. $f = (y)^{1/2}$	F(1) = SQRT(Y(1))
	DO-LOOP F(K) = 0. F(K) = (Y(K) - ATS(K,F,F,2))/F(1)/2.
20. $f = \exp(xa)$	F(1) = EXP(XA(1))
	DO-LOOP F(K) = F(KA)*XA(2)/KA
21. $f = \exp(y)$	F(1) = EXP(Y(1))
	DO-LOOP F(K) = ATT(KA,F,Y,1)
22. $f = \ln(xa)$	F(1) = ALOG(XA(1)) F(2) = XA(2)/XA(1)
	DO-LOOP F(K) = - XA(2)*F(KA)*KB/XA(1)/KA
23. $f = \ln(y)$	F(1) = ALOG(Y(1)) F(2) = Y(2)/Y(1)
	DO-LOOP F(K) = (Y(K) - ATT(KA,Y,F,2))/Y(1)
24. $f = \sin(xa)$ $g = \cos(xa)$	F(1) = SIN(XA(1)) G(1) = COS(XA(1))
	DO-LOOP F(K) = G(KA)*XA(2)/KA G(K) = - F(KA)*XA(2)/KA
25. $f = \sin(y)$ $g = \cos(y)$	F(1) = SIN(Y(1)) G(1) = COS(Y(1))
	DO-LOOP F(K) = ATT(KA,G,Y,1) G(K) = - ATT(KA,F,Y,1)
26. $f = \tan(xa)$	A1(1) = SIN(XA(1)) A2(1) = COS(XA(1)) F(1) = A1(1)/A2(1)
	DO-LOOP A1(K) = A2(KA)*XA(2)/KA A2(K) = - A1(KA)*XA(2)/KA F(K) = (A1(K) - ATS(K,A2,F,2))/A2(1)
27. $f = \tan(y)$	A1(1) = SIN(Y(1)) A2(1) = COS(Y(1)) F(1) = A1(1)/A2(1)
	DO-LOOP A1(K) = ATT(KA,A2,Y,1) A2(K) = - ATT(KA,A1,Y,1) F(K) = (A1(K) - ATS(K,A2,F,2))/A2(1)

FUNCTION	PROGRAM STATEMENTS
28. f = sinh(xa) g = cosh(xa)	F(1) = SINH(XA(1)) G(1) = COSH(XA(1)) DO-LOOP F(K) = G(KA)*XA(2)/KA G(K) = F(KA)*XA(2)/KA
29. f = sinh(y) g = cosh(y)	F(1) = SINH(Y(1)) G(1) = COSH(Y(1)) DO-LOOP F(K) = ATT(KA,G,Y,1) G(K) = ATT(KA,F,Y,1)
30. f = tanh(xa)	A1(1) = SINH(XA(1)) A2(1) = COSH(XA(1)) F(1) = A1(1)/A2(1) DO-LOOP A1(K) = A2(KA)*XA(2)/KA A2(K) = A1(KA)*XA(2)/KA F(K) = (A1(K) - ATS(K,A2,F,2))/A2(1)
31. f = tanh(y)	A1(1) = SINH(Y(1)) A2(1) = COSH(Y(1)) F(1) = A1(1)/A2(1) DO-LOOP A1(K) = ATT(KA,A2,Y,1) A2(K) = ATT(KA,A1,Y,1) F(K) = (A1(K) - ATS(K,A2,F,2))/A2(1)
32. f = $\sin^{-1}$ (xa)	F(1) = ARSIN(XA(1)) A1(1) = COS(F(1)) F(2) = XA(2)/A1(1) A1(2) = - XA(1)*F(2) DO-LOOP F(K) = - ATT(KA,A1,F,2)/A1(1) A1(K) = - F(K)*XA(1) - F(KA)*XA(2)*KB/KA
33. f = $\cos^{-1}$ (xa)	F(1) = ARCOS(XA(1)) A1(1) = SIN(F(1)) F(2) = - XA(2)/A1(1) A1(2) = XA(1)*F(2) DO-LOOP F(K) = ATT(KA,A1,F,2)/A1(1) A1(K) = F(K)*XA(1) + F(KA)*XA(2)*KB/KA
34. f = $\tan^{-1}$ (xa)	F(1) = ATAN(XA(1)) A1(1) = SIN(F(1)) A2(1) = COS(F(1)) F(2) = XA(2)*A2(1)/(A2(1)+XA(1)*A1(1)) A1(2) = A2(1)*F(2) A2(2) = A1(1)*F(2) DO-LOOP F(K) = (XA(2)*A2(KA)-XA(1)*ATT(KA,A1,F,2)- 6 ATT(KA,A2,F,2))/(A2(1)+XA(1)*A1(1)) A1(K) = ATT(KA,A2,F,1) A2(K) = - ATT(KA,A1,F,1)

FUNCTION	PROGRAM STATEMENTS

```
                 -1
35. f = sin   (y)   . . . . . . F(1) = ARSIN(Y(1))
                                A1(1) = COS(F(1))

                              DO-LOOP
                                F(K) = (Y(K) - ATT(KA,A1,F,2))/A1(1)
                                A1(K) = - ATT(KA,Y,F,1)

                 -1
36. f = cos   (y)   . . . . . . F(1) = ARCOS(Y(1))
                                A1(2) = SIN(F(1))

                              DO-LOOP
                                F(K) = - (Y(K) + ATT(KA,A1,F,2))/A1(1)
                                A1(K) = ATT(KA,Y,F,1)

                 -1
37. f = tan   (y)   . . . . . . F(1) = ATAN(Y(1))
                                A1(1) = SIN(F(1))
                                A2(1) = COS(F(1))

                              DO-LOOP
                                F(K) = (ATS(K,Y,A2,2)-Y(1)*ATT(KA,A1,F,2)-
                               6 ATT(KA,A2,F,2))/(A2(1)+Y(1)*A1(1))
                                A1(K) = ATT(KA,A2,F,1)
                                A2(K) = - ATT(KA,A1,F,1)


38. f = si(xa) . . . . It is assumed that the value of
    Sine-integral      the sine-integral, SI(XA) is known.
                                F(1) = SI(XA)
                                A1(1) = SIN(XA(1))
                                A2(1) = COS(XA(1))
                                A3(1) = A1(1)/XA(1)

                              DO-LOOP
                                A1(K) = A2(KA)*XA(2)/KA
                                A2(K) = - A1(KA)*XA(2)/KA
                                A3(K) = (A1(K) - A3(KA)*XA(2))/XA(1)
                                F(K) = A3(KA)*XA(2)/KA
                The above is basically Program II-8 in Chapter II.

39. f = si(y) . . . . . . SI(Y(1)) is assumed to be known.
    Sine-integral               F(1) = SI(Y(1))
                                A1(1) = SIN(Y(1))
                                A2(1) = COS(Y(1))
                                A3(1) = A1(1)/Y(1)

                              DO-LOOP
                                A1(K) = ATT(KA,A2,Y,1)
                                A2(K) = - ATT(KA,A1,Y,1)
                                A3(K) = (A1(K) - ATS(K,Y,A3,2))/Y(1)
                                F(K) = ATT(KA,A3,Y,1)
                The above is basically Program II-9 in Chapter II.

40. f = erf(xa)  . . . ERF(XA(1)) is assumed to be known.
    Error function              F(1) = ERF(XA(1))
                                A1(1) = EXP(-XA(1)*XA(1))
                                C = 2./SQRT(3.14159)
                                F(2) = C*A1(1)*XA(2)
                                A1(2) = - A1(1)*XA(1)*XA(2)*2.

                              DO-LOOP
                                A1(K) = - (A1(KA)*XA(1) + A1(KB)*XA(2))*
                               6 XA(2)*2./KA
                                F(K) = C*A1(KA)*XA(2)/KA
                The above is basically Program II-6 in Chapter II.
```

FUNCTION	PROGRAM STATEMENTS
41. f = erf(y) . . . . Error function	ERF(Y(1)) is assumed to be known.

```
        F(1) = ERF(Y(1))
        A1(1) = - Y(1)*Y(1)
        A2(1) = EXP(A1(1))
        C = 2./SQRT(3.14159)

     DO-LOOP
        A1(K) = - ATS(K,Y,Y,1)
        A2(K) = ATT(KA,A2,A1,1)
        F(K) = C*ATT(KA,A2,Y,1)
```

The above is basically Program II-7 in Chapter II.

42. f = Bessel function . . See either Program II-4, or Program II-5 in Chapter II. These programs are much too complicated for this Appendix.